Notes on Representation Theory

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About This Notes

Note. This is under construction!

This notes mostly amount to an amalgamation of thoughts and ideas I came across when studding the representation theory of groups. Many of the comments in here don't have much *mathematical* value per say – they are better understood as *philosophical* considerations on the topic at hand. The focus of this notes is the representation theory of Lie groups and algebras, but we'll dive into some other topics too. We'll assume basic knowledge of abstract algebra, group theory and differential geometry.

Lengthy proves are favored as opposed to collections of smaller lemmas, since we want to emphasize the relevant results.

Contents

Chapter 1

Groups & Actions

> A group is a groupoid with a single element
>
> *Paolo Aluffi*
> *Algebra: Chapter 0*

Group theory has a special place in abstract algebra: groups are simple and elegant, yet interesting enough to study – unlike monoids or magmas. The interesting thing about groups though is they transcend abstract algebra. What I mean by that is that in some sense groups are much more fundamental than what their simple algebraic structure may imply. Groups are algebraic incarnation of *actions* and *symmetries*.

Definition. If $\mathcal{C}$ is a category and a is an object of $\mathcal{C}$, an *action* of G in a is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(a)$$

If ρ is injective ρ is called a faithful action.

Just like men – and women – a group is known by it's actions. Indeed, Caley's theorem establishes that every group is isomorphic to a subgroup of a permutation group. In other words, every group G is characterized by a set X and a faithful action of G in X, namely $X = G$ and

$$\begin{aligned} \rho : G &\longrightarrow S_X \\ g &\longmapsto \rho(g) : X \longrightarrow X \\ &\qquad\qquad\quad x \longmapsto g \cdot x \end{aligned}$$

This implies a group is essentially a group action in the category **Set** of sets. In the abstract *categorical* terms of the last definition, Caley's theorem amounts to the epigraph if this chapter – which is a fancy way of saying *a group is a thing that acts on some other* arbitrary *thing*.

We are usually interested in actions of a group G with some extra structure that *respect the extra structure of* G in some sense or another, such as (smooth) isometric actions of a Lie group over Riemannian manifolds. Notice that given a (connected) Riemannian manifold M, $\mathrm{Iso}(M)$ is a Lie group under composition – this is apparently called *the Myners-Steenrod theorem* – so saying *"$\rho : G \longrightarrow \mathrm{Iso}(M)$ is a smooth map"* actually makes sense.

Note. Clearly, every manifold is connected – otherwise we're really talking about multiple manifolds.

We'll look into some other examples of this when we discuss continuous representations of compact groups in chapter 2 and smooth representations of Lie groups in chapter 3, but before we get into that let's review some basic concepts in representation theory.

1.1 Representations

The single most well understood category in the entirety of mathematics is $\mathbb{C}$**-Vect**. Hence actions on $\mathbb{C}$**-Vect** are a natural starting point for understanding the behavior of a given group.

Definition 1.1.1. A complex vector space V equipped with an action $\rho : G \longrightarrow \mathrm{GL}(V)$ is called a *representation* of G – or alternatively a *linear action of* G. It's common to denote $\rho(g)$ by simply g when ρ can be inferred from the surrounding context.

Example 1.1.1. Given a group G, the group algebra $\mathbb{C}[G]$ equipped with the left multiplication

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(\mathbb{C}[G]) \\ g &\longmapsto \rho(g) : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \\ &\qquad\qquad\qquad x \longmapsto g \cdot x \end{aligned}$$

is a representation of G.

For finite groups, $\mathbb{C}[G]$ is identified with the space $\mathbb{C}^G$ of functions $G \longrightarrow \mathbb{C}$, which is called *the regular representation of* G. Under this identification,

$$(g \cdot f)(h) = f(g^{-1}h)$$

Example 1.1.2. Given two representations V and W of G, $V \oplus W$ and $V \otimes W$ are both representations of G.

Example 1.1.3. Given a representation V of G and a representation W of H, the space $V \boxtimes W = V \otimes W$ with

$$(g, h)\, v \otimes w = gv \otimes hw$$

is a representation of $G \times H$.

Example 1.1.4. Given a representation V of G and a subgroup $H \subseteq G$, the space $\mathrm{Res}_H^G V = V$ is a representation of H – where the action of H is given by restricting $\rho : G \longrightarrow \mathrm{GL}(V)$ to H.

In particular, if V and W are representations of G then $\mathrm{Res}_{\Delta G \times G}^{G \times G} V \boxtimes W$ is identified with the representation $V \otimes W$ of G – where $\Delta G \times G$ is the subgroup $\{(g, g) : g \in G\} \subseteq G \times G$.

As you might have guessed, *representation theory* (of groups) is the study of *representations* of groups. The general goal of representation theory is to extract as much information about a given group G as possible from it's representations. It's also worth noting that understanding the relationship between representations is an integral part of representation theory. This brings us to the following definitions.

Definition 1.1.2. Given a group G and two representations V and W of G, a linear map

$$T : V \longrightarrow W$$

such that

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[T]{} & W \end{array}$$

for all $g \in G$ is called an *intertwing* operator or an *intertwiner*. Intertwing operators can be thought-of as *maps that preserve the action of* G.

Definition 1.1.3. Given a group G and a representation V of G, a subspace $W \subseteq V$ such that $GW \subseteq W$ is called a *subrepresentation* of V. If the V admits no proper subrepresentations – subrepresentations other than 0 and V – then V is called an *irreducible* representation.

This definitions naturally induce a category $\mathbf{Rep}(G)$, whose objects are representations of G and whose morphisms are intertwining operators. It turns out much of the structure of a group G can be reconstructed from $\mathbf{Rep}(G)$. In fact, a multitude of reconstruction theorems from the likes of Lev Pontryagin and Tadao Tannaka establish that, in certain contexts, the entirety of G can be reconstructed from the category $\mathbf{FinRep}(G)$ of finite-dimensional representations of G.

This hopefully establishes that *representation theory is useful*, but it also poses a problem. The reconstruction theorems require us to understand the whole of $\mathbf{Rep}(G)$ – or at least a large chunk of it. In other words, understanding individual representations won't get us anywhere, we need to study the collective behavior of *all* representations of group to be able to extract useful information from them.

Hence the classical problem in representation theory is classifying all representations of a given group G up to isomorphism. This turns out to be hard. However, the problem of classifying the finite-dimensional representations of a finite group G is a solved one. This will be the focus of our next section.

1.2 Finite-dimensional Representations of Finite Groups

The theory of finite-dimensional representations of finite groups usually serves as a first introduction to representation theory, while the simplicity and elegance of some of it's core arguments serve as inspiration for the theory used in more complicated settings.

The first instrumental piece of this theory is…

Lemma 1.2.1 (Schur). *If V and W are irreducible representations of a group G and $T : V \longrightarrow W$ is an intertwining operator then T is either 0 or an isomorphism of representations.*

Proof. It suffices to note that $\ker T \subseteq V$ and $\operatorname{im} T \subseteq W$ are both subrepresentations. Hence T is either bijective or 0. ■

The key to solving the problem of the classification of finite-dimensional representations of finite groups lies in an innocent observation…

Theorem 1.2.1 (Maschke). *Every finite-dimensional representation of a finite group is isomorphic to the direct sum of irreducible representations.*

We'll go over a proof of Maschke's theorem in the following chapter. For now, simply note that Maschke's theorem implies $\mathbf{FinRep}(G)$ is a semisimple category. In other words, understanding the irreducible representations of G is enough for reconstructing the entirety of $\mathbf{FinRep}(G)$. This observations, combined with something called *character theory* are enough to *completely annihilate* and *utterly destroy* our initial classification problem.

Unfortunately, not every group is finite. Our next question is…what happens if G is infinite? We begin our inquire by investigating the case of compact groups, which provides us a welcoming introduction to the world of topological groups and their representations. Many of the details omitted in this section will be covered in the next chapter. Please refer to [2], [9] and [4] for further details.

Chapter 2

Continuous Representations of Compact Groups

The following chapter is generally based on the third chapter of *A Journey Through Representation Theory: From Finite Groups to Quivers via Algebras* [2].

The algebraic theory of finite groups and their representations is quite rich, but it's infinite counterpart is generally lacking in comparison. Infinite groups are complex beasts on their own, and one usually has to endow them with geometric structure to get interesting results. The simplest geometric structure in town is topology, so one naturally pays special attention to topological groups – i.e. group objects in the category **Top** of topological spaces.

$$\begin{array}{ccc} \mathbf{GrpTop} & \longrightarrow & \mathbf{Grp} \\ \big\downarrow & & \big\downarrow \\ \mathbf{Top} & \longrightarrow & \mathbf{Set} \end{array}$$

Just as one only considers continuous group homomorphisms when dealing with topological groups, given a topological group G it's usual practice to ignore representations that *do not respect the topological structure of* G. But what is that supposed to mean?

Definition. A representation V of a topological group G is called *continuous* if V is a (Hausdorff) topological vector space and the map

$$(g, v) \longmapsto gv$$

is continuous.

An alternative formulation I've seen around is something along the lines of *a finite-dimensional representation V of G over $\mathbb{R}$ or $\mathbb{C}$ is continuous if*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is continuous. Note that $\mathrm{GL}(V)$ is a topological group, since $\mathrm{GL}(V)$ is one of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. The former definition is, of course, much more general, but I still find the latter more intuitive.

Likewise, we only consider continuous intertwining operators between continuous representations of G, and closed subrepresentations of continuous representations of G. Note that in this context, the phrase *V is irreducible* means *the only* closed *subrepresentations of V are* 0 *and* V.

I suppose this makes sense from the *categorical* perspective, but endowing groups and representations with a topological structure just for the sake of it isn't productive at all. Why is any of this useful?

Well, the point of all this is that the topology of G and it's representations allows us to reproduce many of the results of the theory of representations of finite groups, as we'll establish in the following. The parallels between representations of finite groups and representations of topological groups come mainly in the form of *tools derived from the Haar measure on compact groups* and the study of *unitary representations*. The closest thing in topology to finite groups are compact groups, so those will be the focus of the following chapter.

2.1 The Haar Measure on Compact Groups

As one would expect, the study of representations of finite groups is greatly facilitated by the fact that *finite groups are finite*. In practice this means things like

$$\sum_{g \in G} f(g)$$

are well defined for finite G. This is enormously helpful when building G-invariant construction on representations of G, since we can just take *any* construction on G and make it invariant by averaging over G.

The classic example of this is the existence of a G-invariant (positive definite) Hermitian inner product on a representation V of G over $\mathbb{C}$. Given *any* Hermitian product H on V, one can define

$$\langle v, u \rangle = \frac{1}{|G|} \sum_{g \in G} H(gv, gu)$$

and obtain a G-invariant Hermitian product on V. Now given a proper subrepresentation $W \subsetneq V$, notice $U = W^{\perp}$ is a subrepresentation of V. It then follows from the finite induction principle that Maschke's theorem holds. Ideally, we would like to extend some of this tools to the study of locally compact groups.

It turns out every Hausdorff locally compact group G admits a non-trivial G-invariant – either by left or right translations – Borel measure, and this measure is unique up to multiplication by a positive scalar. This measure is quite well-behaved too, in particular it is locally finite, regular and $\mu(U) > 0$ for every non-empty open subset $U \subseteq G$. The fact that G is required to be Hausdorff may seem like a huge limitation, but remember every T_0 topological group is actually Hausdorff.

This is called the – either *left* or *right* – *Haar measure* of G, and it allows us to reproduce some of the averaging arguments used for finite groups by replacing sums with integrals

$$\sum_{g \in G} f(g) \rightsquigarrow \int_G f(g) \, \mathrm{d}g$$

Note. From now on we'll denote the Haar measure of a Hausdorff locally compact group G by μ.

Example 2.1.1. Let G be a discrete group and μ be it's left Haar measure. Then $\{e\}$ is an open subset of G and therefore $\{e\}$ is measurable. Let $\lambda = \mu(\{e\})$.

Since $\{e\}$ is non-empty, $\lambda > 0$. Furthermore, since μ is left-invariant, $\mu(\{g\}) = \mu(\{e\}) = \lambda$ for each $g \in G$. Hence given a finite subset $X \subseteq G$

$$\mu(X) = \sum_{g \in X} \mu(\{g\}) = \sum_{g \in X} \lambda = \lambda \, |X|$$

If $X \subseteq G$ is a countably infinite subset then

$$\mu(X) = \sum_{g \in X} \mu(\{g\}) = \sum_{g \in X} \lambda = \infty$$

If $Y \subseteq G$ is uncountable, then there exists a countably infinite subset $X \subseteq Y$. Hence

$$\mu(Y) \geqslant \mu(X) = \infty$$

and therefore $\mu(Y) = \infty$. In conclusion, μ is a scalar multiple of the counting measure.

The left Haar measure of a locally compact group G isn't necessarily proportional to the right Haar measure of G, but this is the case for compact groups. Furthermore, if G is compact then G has finite measure and therefore the Haar measure of G can be normalized so that $\mu(G) = 1$. We won't normalize the Haar measure in this notes though, since we want to emphasize the parallels with finite-dimensional representations of finite groups.

The proof that every (Hausdorff) locally compact group admits a Haar measure is somewhat involved. A special class of locally compact groups are the Lie groups, also known as *groups that are also smooth manifolds* – i.e. group objects in the category **Diff** of smooth manifolds. It's much easier to show that every Lie group admits a Haar measure, precisely because Lie groups are orientable manifolds and therefore they admit a (G-invariant) volume form, as we'll establish in the following.

Note. As manifolds, Lie groups are locally Euclidean. Hence Lie groups are *locally locally compact*, i.e. locally compact.

2.1.1 Differential Forms

As previously mentioned, differential forms allow us to easily prove that every Lie group admits a Haar measure. We'll go over the definition of a differential form in a smooth manifold before proceeding to the proof mentioned above. The following comments are based on Marcos Alexandrino's notes on Riemannian Geometry [1].

Definition 2.1.1. Let M be an n-dimensional smooth manifold. Given $p \in G$, the tangent T_pM space at p is an n-dimensional real vector space. Now consider the real vector space $\wedge^k(T_pM)^*$ of k-linear functionals $\omega_p : T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R}$ such that

$$\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \cdot \omega_p(v_1, \ldots, v_k)$$

for all $\sigma \in S_k$.

Since $\left\{ \left.\frac{\partial}{\partial x_i}\right|_p \right\}_i$ is a basis for T_pM,

$$\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : i_1 < \cdots < i_n\}$$

is a is a basis for $\wedge^k(T_pM)^*$ – where $\{\mathrm{d}x_i\}_i$ is the dual basis of $\left\{ \left.\frac{\partial}{\partial x_i}\right|_p \right\}_i$. This implies

$$\dim \wedge^k(T_pM)^* = \binom{n}{k}$$

This construction induces a vector bundle

$$\begin{array}{c} \coprod_{p \in M} \wedge^k(T_pM)^* \\ \big\downarrow \\ M \end{array}$$

commonly denoted by $\wedge^k T^*M$.

A (smooth) section of $\wedge^k T^*M$ – i.e. a smooth function $\omega : M \longrightarrow \wedge^k T^*M$ that takes each $p \in M$ to some $\omega_p \in \wedge^k(T_pM)^*$ – is called a *differential form of degree k*. Note that $\dim \wedge^n T_p^*M = 1$. Hence differential forms of degree n are called *differential forms of maximal degree*. Nowhere-vanishing – non-zero at every point – differential forms of maximal degree are called *volume forms*.

The set of all differential forms of degree k is usually denoted by $\Omega^k(M)$.

Notice $\Omega^k(M)$ is a $C^\infty(M)$-module where

$$(f \cdot \omega)_p = f(p) \cdot \omega_p$$

Moreover, an element $\omega \in \Omega^k(M)$ can be thought-of as a k-linear alternating functional

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $\mathfrak{X}(M)$ is the C^∞-module of (smooth) vector fields over M and

$$(\omega(V^1, \ldots, V^k))(p) = w_p(V^1_p, \ldots, V^k_p)$$

Why is this useful to us though? Well, besides various applications on differential geometry – which I'm completely unaware of – given a Lie group G, a differential form of maximal degree $\omega \in \Omega^n(G)$ can be integrated over of G. In other words,

$$\int_G \omega \in [-\infty, \infty]$$

is a thing that exists and we can use this to build a measure over G. I'll spear the reader from the boring details of the construction of the integral of a differential form, all we need to know at the moment is that integration of volume forms is something generally well behaved. Please refer to [1] for more information on this topic.

Theorem 2.1.1. *Every Lie group G admits a left Haar measure.*

Proof. Suppose $\dim G = n$. Given $g \in G$, the left translations $\ell_{g^{-1}} : G \longrightarrow G$ by g^{-1} is a diffeomorphism that takes g to e. Consider the induced isomorphism

$$\ell^*_{g^{-1}} : \wedge^n T^*_e G \longrightarrow \wedge^n T^*_g G$$

given by $\wedge^n (d\ell_{g^{-1}})^*_g$ – where $(d\ell_{g^{-1}})^*_g : T^*_e G \longrightarrow T^*_g G$ is the dual of $(d\ell_{g^{-1}})_g : T_g G \longrightarrow T_e G$.

Let $\omega_e = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \wedge^n T^*_e G$. Then the map

$$\begin{aligned} \omega : G &\longrightarrow \wedge^n T^* G \\ g &\longmapsto \ell^*_{g^{-1}}(\omega_e) \end{aligned}$$

is a volume form. We claim ω is left-invariant too. Indeed,

$$\begin{aligned} (\ell^*_g \omega)_h &= \wedge^n (d\ell_g)^*_h \, \omega_{gh} \\ &= \wedge^n (d\ell_g)^*_h \wedge^n (d\ell_{h^{-1}g^{-1}})^*_{gh} \, \omega_e \\ &= \wedge^n ((d\ell_g)^*_h (d\ell_{h^{-1}g^{-1}})^*_{gh}) \, \omega_e \\ &= \wedge^n ((d\ell_{h^{-1}g^{-1}})_{gh} (d\ell_g)_h)^* \, \omega_e \\ \text{(chain rule to the rescue)} &= \wedge^n (d\ell_{h^{-1}})^*_h \, \omega_e \\ &= \ell^*_{h^{-1}}(\omega_e) \\ &= \omega_h \\ \therefore \ell^*_g \omega &= \omega \end{aligned}$$

Suppose without any loss of generality that

$$\int_G \omega \geqslant 0,$$

so that

$$\mu(K) = \int_K \omega \geqslant 0$$

for each compact subset $K \subseteq G$.

Clearly, $\mu(K) < \infty$, and it's easy to check that $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a countably additive monotonic function too. Moreover,

$$\mu(gK) = \int_{\ell_g(K)} \omega = \int_K \ell_g^* \omega = \int_K \omega = \mu(K)$$

and therefore $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a left-invariant pre-measure.

By Caratheodory's extension theorem, μ can be extended to every Borel subset of G, yielding a left-invariant Borel measure $\mu : \mathfrak{B}(G) \longrightarrow [0, \infty]$ – see San Martin's construction of the Haar measure [7] for further details. Furthermore, since ω is nowhere-vanishing, μ is non-trivial. We are done. ■

Note. Notice we've also proved G is orientable – i.e. G admits a volume form.

The proof that every Lie group admits a right Haar measure is essentially the same – just replace ℓ with $\imath$ everywhere. Let's assume that every locally compact group admits a Haar measure and proceed to some applications of this fact.

2.2 Unitary Representations

Unitary representations play a special role in the representation theory of compact groups. In particular, there's a formulation o Schur's lemma that holds for unitary representations, which allows us to reproduce many of the results of the theory of representations of finite groups in the context of compact groups.

As one would expect, a unitary representation of a topological group G is a representation in which every element $g \in G$ acts *unitarily* – i.e. g is a unitary operator. In other words...

Definition 2.2.1. A continuous representation V of a topological group G is called *unitary* if V is a Hilbert space and

$$\langle v, w \rangle = \langle gv, gw \rangle$$

for all $g \in G$.

Example 2.2.1. If G is a compact group then the space $L^2(G)$ of functions $f : G \longrightarrow \mathbb{C}$ such that

$$\int_G |f(g)|^2 \, \mathrm{d}g$$

exists is a unitary representation of G, where the action of $g \in G$ is given by

$$(g \cdot f)(h) = f(g^{-1}h)$$

and

$$\langle f_1, f_2 \rangle = \frac{1}{\mu(G)} \int_G f_1(g) \overline{f_2(g)} \, \mathrm{d}g$$

This is called the regular representation of G.

When G is finite and discrete the integral over G is just the regular summation and $L^2(G)$ is the space of all complex-valued functions on G – also known as $\mathbb{C}[G]$. In other words, this is a strict generalization of the regular representation of finite groups.

I'll restate some of the results presented in *A Journey Through Representation Theory: From Finite Groups to Quivers via Algebras* [2] in here for the sake completion and to highlight some of the parallels between representations of finite groups and unitary representations of compact groups. For the missing proofs see [2].

As we'll establish in the following, unitary representations are in many regards the compact analogous of finite-dimensional representations of finite groups.

Lemma 2.2.1. *Every non-zero unitary representation V of a compact group G has a non-zero finite-dimensional (closed) subrepresentations.*

Note. This proof is quite involved so buckle your seat-belts.

Proof. Given $u \in V$ with $\|u\| = 1$, definite

$$\begin{aligned} Q : V &\longrightarrow V \\ v &\longmapsto \int_G gTg^{-1}v \, \mathrm{d}g \end{aligned}$$

where

$$\begin{aligned} T : V &\longrightarrow V \\ v &\longmapsto \langle v, u \rangle u \end{aligned}$$

We want to establish that $\ker Q - \lambda \, \mathrm{Id}$ is a non-zero finite-dimensional subrepresentation of V for some $\lambda > 0$.

Notice Q is a self-adjoint intertwining operator. Furthermore

$$\begin{aligned} \langle v, Qv \rangle &= \left\langle v, \int_G \langle g^{-1}v, u \rangle gu \, \mathrm{d}g \right\rangle \\ &= \int_G \langle v, gu \rangle \overline{\langle v, gu \rangle} \, \mathrm{d}g \\ &= \int_G |\langle v, gu \rangle|^2 \, \mathrm{d}g \\ &\geqslant 0 \end{aligned}$$

so Q is semipositive. We claim Q is compact too.

Indeed, it follows from the Cauchy-Schwartz inequality that T is bounded. Moreover,

$$\dim \operatorname{im} T = \dim \langle u \rangle = 1$$

so gTg^{-1} is a bounded operator with 1-dimensional image for all $g \in G$. This implies

$$Q = \int_G gTg^{-1} \, \mathrm{d}g$$

is compact – since the ideal $\mathcal{C}(V) \subseteq \mathcal{B}(V)$ of all compact operators $V \longrightarrow V$ is the closure of the ideal $\mathcal{F}(V) \subseteq \mathcal{B}(V)$ of bounded operators $V \longrightarrow V$ with finite-dimensional image.

In other words, Q is a compact semipositive self-adjoint intertwining operator. Hence

$$\lambda = \sup_{\|v\|=1} \langle v, Qv \rangle$$

is either 0 or an eigenvalue of T. Notice

$$\langle u, Qu \rangle = \int_G |\langle u, gu \rangle|^2 \, \mathrm{d}g > 0$$

since

$$g \longmapsto |\langle u, gu \rangle|^2$$

is a continuous map that takes every element of G to a non-negative real number and

$$|\langle u, u \rangle|^2 = 1 > 0$$

This implies $\lambda > 0$, so λ is a positive eigenvalue of Q. Let $W = \ker Q - \lambda \, \mathrm{Id}$. Notice W is a (closed) subrepresentation of V, since it is the kernel of a continuous intertwining operator. Furthermore, W is non-zero since it is the eigenspace of T associated to $\lambda > 0$.

In conclusion, since Q is compact and self-adjoint and $\lambda \neq 0$, it follows from the spectral theorem that W is finite-dimensional. Finally, W is a finite-dimensional (closed) subrepresentation of V! ∎

Corollary 2.2.1. *Every irreducible unitary representation of a compact group is finite-dimensional.*

Corollary 2.2.2 (Schur)**.** *If V as an irreducible unitary representation of a compact group G and $T : V \longrightarrow V$ is a continuous intertwining operator, then there exists $\lambda \in \mathbb{C}$ such that $T = \lambda \,\mathrm{Id}$.*

Proof. Since V is unitary, V is finite-dimensional. It then follows that T has at least one eigenvalue, so $\ker T - \lambda \,\mathrm{Id} \neq 0$ for some $\lambda \in \mathbb{C}$.

Notice $T - \lambda \,\mathrm{Id}$ is a continuous intertwiner. This implies that $\ker T - \lambda \,\mathrm{Id}$ is a (closed) subrepresentation of V. Since $\ker T - \lambda \,\mathrm{Id} \neq 0$, $\ker T - \lambda \,\mathrm{Id} = V$ and therefore $T = \lambda \,\mathrm{Id}$. ■

Corollary 2.2.3. *Let V and W be two irreducible unitary representations of a compact group G and $T : V \longrightarrow W$ be a continuous intertwining operator. Then either $T = 0$ or there exists $\lambda > 0$ such that λT is an isometry.*

Proof. Suppose $T \neq 0$. It follows from Schur's lemma that

$$T^*T = \mu \,\mathrm{Id}$$

for some $\mu \in \mathbb{C}$. We want to establish that μ is a positive real number.

Since μ is an eigenvalue of the bounded semipositive self-adjoint operator T^*T, μ is a non-negative real number. Suppose $\mu = 0$. Then $T^*T = 0$ and therefore T is a not invertible. ↯

Hence $\mu \neq 0$ and therefore $\mu > 0$. Let $\lambda = \frac{1}{\sqrt{\mu}}$ and $U = \lambda T$. Then

$$\begin{aligned} U^*U &= \frac{1}{\sqrt{\mu}} T^* \frac{1}{\sqrt{\mu}} T \\ &= \frac{1}{|\mu|}\, \mu \,\mathrm{Id} \\ &= \mathrm{Id} \end{aligned}$$

In other words, $U = \lambda T$ is an intertwining isometry. ■

Corollary 2.2.4. *Every irreducible unitary representation of a compact Abelian group is* 1*-dimensional.*

Proof. It suffices to note that $g : V \longrightarrow V$ is a continuous intertwiner for each $g \in G$, precisely because G is Abelian. ■

Theorem 2.2.1. *Every irreducible continuous representation of a compact group G is isomorphic to a subrepresentation of the regular representation $L^2(G)$ of G.*

Corollary 2.2.5. *Every irreducible continuous representation of a compact group G is isomorphic to a unitary representation.*

Proof. It suffices to note that $L^2(G)$ is a unitary representation. ■

It turns out Schur's lemma, corollary 2.2.4 and corollary 2.2.5 all hold for locally compact groups too. This is essential for establishing the connection between Pontryagin's duality and representation theory. For more information on the topic please refer to *Fourier Analysis on Number Fields* [3].

Corollary 2.2.6. *Every irreducible continuous representation of a compact group G is finite-dimensional.*

Theorem 2.2.2. *Every unitary representation of a compact group G is completely reducible. In other words, given a unitary representation V of G and a (closed) subrepresentation $W \subseteq V$, there exists a (closed) subrepresentation $U \subseteq V$ such that*

$$V \cong W \oplus U$$

Proof. Consider $U = W^\perp$. Note U is a closed subspace of V. Furthermore, given $u \in U$, $w \in W$ and $g \in G$, it follows from the fact that $g^{-1}w \in W$ that

$$\langle gu, w\rangle = \langle u, g^{-1}w\rangle = 0$$

This implies U is G-invariant – i.e. a subrepresentation of V. In conclusion, U is a (closed) subrepresentation of V and

$$V \cong W \oplus U$$

■

Corollary 2.2.7 (Maschke)**.** *Every finite-dimensional continuous representation of a compact group G is semisimple.*

Note. The proof of Maschke's theorem for compact groups is precisely the same as the proof for finite groups!

Proof. Let V be a finite-dimensional continuous representation of G. Given a Hermitian inner product H, consider

$$\langle v, u\rangle = \frac{1}{\mu(G)} \int_G H(gv, gh)\, \mathrm{d}g$$

Our goal is to show that V is a unitary representation under $\langle\,,\rangle$.

Note $\langle\,,\rangle$ is a G-invariant inner product on V. We want to establish that the topology of V is the topology induced by $\langle\,,\rangle$.

Since $\dim V < \infty$, V is isomorphic to $\mathbb{C}^n$ as a topological vector space – where $n = \dim V$. Hence any two metrics on V derived from norms are equivalent. In particular, the metric derived from $\langle\,,\rangle$ and the Euclidean distance are equivalent.

This implies the topology of V is the topology induced by $\langle\,,\rangle$. Furthermore, since the Euclidean space is complete under the Euclidean distance, V is complete under the distance induced by $\langle\,,\rangle$.

So V is a Hilbert space under $\langle\,,\rangle$ and $\langle\,,\rangle$ is G-invariant. In other words, V is a unitary representation of G. Hence V is completely reducible. In conclusion, since V is finite-dimensional and completely reducible, V is semisimple. ■

Note that for finite G, μ is simply the counting measure and the integral is the standard summation, so this construction is compatible with the construction used for finite groups.

2.3 The Peter-Weyl Theorem & Character Theory

Another important aspect of the theory of representations of compact groups is something called *character theory.* Given a finite-dimensional representation V of a compact group G, consider the function

$$\begin{aligned} \chi_V : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \mathrm{Tr}(g\restriction_V) \end{aligned}$$

This is called the *character of* V. If V is irreducible, χ_V is called an irreducible character. Clearly, the characters of two isomorphic representations coincide, since the trace of a matrix is invariant under changes of basis. Hence characters are invariants of representations. The same argument can be used to show that characters of representations of a group G are constant in the conjugacy classes of G – i.e. characters are *class functions.* Moreover, it's easy to check that $\chi_{V\oplus W} = \chi_V + \chi_W$.

Why are we doing all this though? Well, it turns out characters of finite groups are not only invariants, they are *perfect invariants.* In other words, a finite-dimensional representation of a finite group is completely determined by it's character – i.e. if V and W are finite-dimensional representations of a finite group G and $\chi_V = \chi_W$ then $V \cong W$. This is a remarkably powerful result, which follows immediately from the fact that…

Theorem 2.3.1. *The characters of irreducible representations of a finite group G are orthonormal in $L^2(G)$.*

Hence…

Corollary 2.3.1. *A finite-dimensional representation of a finite group G is completely determined by it's character. In other words, the ring $R(G)$ of virtual representations of G is a free $\mathbb{Z}$-module.*

Proof. Given a finite-dimensional representation V of G, it follows from Maschke's theorem that

$$V \cong \bigoplus_{i=1}^{n} V_i$$

for some irreducible representations $V_1, V_2, \dots V_n$ of G. This implies $\chi_V = \chi_{V_1} + \cdots + \chi_{V_n}$ is a linear combination of the irreducible characters of G.

Since the irreducible characters of G are orthonormal, they for a basis of the subspace of functions $G \longrightarrow \mathbb{C}$ spanned by the irreducible characters themselves. This implies that χ_V can be uniquely expressed as a linear combination of irreducible characters of G.

Now let W be a finite-dimensional representation of G such that $\chi_W = \chi_V$. Suppose

$$W \cong \bigoplus_{i=1}^{m} W_i$$

for some irreducible representations $W_1, W_2, \dots, W_m$ of G. It then follow that

$$\chi_{V_1} + \cdots + \chi_{V_n} = \chi_V = \chi_W = \chi_{W_1} + \cdots + \chi_{W_m},$$

so $W_i = V_i$. This establishes that $V \cong W$. ∎

This comes in handy for verifying that a representation is irreducible, since…

Corollary 2.3.2. *A finite-dimensional representation V of G is irreducible if and only if*

$$\langle \chi_V, \chi_V \rangle = 1$$

Corollary 2.3.2 can be used to show that…

Theorem 2.3.2. *If V is the set of all irreducible representations of G up to isomorphism then*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \widehat{G}} \dim V \cdot V$$

Another important consequence of theorem 2.3.1 is…

Theorem 2.3.3. *The irreducible characters of a finite group G form an orthonormal basis of the space $\mathcal{C}(G)$ of class functions $G \longrightarrow \mathbb{C}$.*

For a proof of theorem 2.3.3 see [9]. This implies that the number of irreducible representations of a finite group G – up to isomorphism – is precisely the number of conjugacy classes of G. The goal of this section is to generalize this results to compact groups. The case of compact groups is more involved, since we lack some of the tools used for finite groups. Nevertheless, this results can be generalized by using something called *matrix coefficients*.

Definition 2.3.1. Given a unitary representation V of a compact group G and $v, w \in V$. Consider

$$\begin{aligned} f_{v,w} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle gv, w \rangle \end{aligned}$$

This is called a *matrix coefficient* of V.

This is particularly useful for computing the character of finite-dimensional representations, since if $g = (a_{ij})_{ij}$ in an orthonormal basis $\{e_1, \ldots, e_n\}$ then $f_{e_i,e_j}(g) = a_{ij}$. Hence

$$\chi_V(g) = \mathrm{Tr}(g\restriction_V) = \sum_{i=1}^{n} f_{e_i,e_i}(g)$$

Using matrix coefficients, it's easy enough to show that theorem 2.3.1, corollary 2.3.1 and corollary 2.3.2 all hold for compact groups [2], but ideally we would also like to show that theorem 2.3.2 holds. Moreover, we would like to show that the irreducible characters of a compact G span the subspace $\mathcal{C}(G) \subseteq L^2(G)$ of class functions – the word *span* is doing some legwork here, we'll get to the precise formulation of this. Proving this requires *the Peter-Weyl theorem.* This theorem may not seem particularly interesting on it's own, but it is essential for generalizing theorem 2.3.2 and theorem 2.3.3.

Theorem 2.3.4 (Peter-Weyl)**.** *The matrix coefficients of all irreducible representations of compact group G span a dense subspace in $L^2(G)$.*

Lemma 2.3.1. *The map*

$$\begin{aligned}\Phi : \overline{\bigoplus_{V \in \widehat{G}} V^* \boxtimes V} &\longrightarrow L^2(G)\\ e_i^* \otimes e_j &\longmapsto f_{e_j,e_i}\end{aligned}$$

is an isomorphism of $G \times G$-representations where the action of $G \times G$ in $L^2(G)$ is given by

$$((g,h)f)(k) = f(g^{-1}kh)$$

Proof. It's clear that Φ is linear, and it follows from the Peter-Weyl theorem that Φ is also surjective. To see that Φ is an intertwiner, it suffices to observe that

$$\begin{aligned}((g,h)\phi(e_i^* \otimes e_j))(k) &= ((g,h)f_{e_j,e_i})(k)\\ &= f_{e_j,e_i}(g^{-1}kh)\\ &= \langle g^{-1}khe_j, e_i\rangle\\ &= \langle khe_j, ge_i\rangle\\ &= f_{he_j,ge_i}(k)\\ &= (\phi(ge_i^* \otimes he_j))(k)\\ &= (\phi((g,h)e_i^* \otimes e_j))(k)\\ &= (g,h)\phi(v \otimes w) = \phi((g,h)v \otimes w)\end{aligned}$$

Now given $V \in \widehat{G}$, notice $\|\chi_{V^* \boxtimes V}\| = \|\chi_{V^*}\| \cdot \|\chi_V\| = 1$. It then follow from corollary 2.3.2 that $V^* \boxtimes V$ is irreducible. Hence $\ker \Phi\restriction_V$ is either 0 or V. Since $\Phi(e_1^* \otimes e_1) = f_{e_1,e_1} \neq 0$, $\ker \Phi\restriction_V = 0$. This implies Φ is injective.

In conclusion, Φ is an isomorphism of representations. ■

Again, this results aren't that appealing on their own. What we're really interested in is...

Corollary 2.3.3 (Peter-Weyl)**.**

$$L^2(G) \cong \overline{\bigoplus_{V \in \widehat{G}} \dim V \cdot V}$$

Proof. Notice $G \times \{e\} \cong G$ as topological groups. On the one hand, $\operatorname{Res}^{G\times G}_{G\times\{e\}} L^2(G)$ is precisely the regular representation $L^2(G)$. On the other hand, given $V \in \widehat{G}$ it's easy to check that

$$\begin{aligned}
\operatorname{Res}^{G\times G}_{G\times\{e\}} V^* \boxtimes V &\cong V^* \otimes \bigoplus_{i=1}^{\dim V} \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V} V^* \otimes \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V^*} V^* \\
&= \dim V^* \cdot V^*
\end{aligned}$$

Moreover, since V is irreducible, V^* is irreducible. This implies

$$\operatorname{Res}^{G\times G}_{G\times\{e\}} \overline{\bigoplus_{V\in\widehat{G}} V^* \boxtimes V} \cong \overline{\bigoplus_{V\in\widehat{G}} \dim V \cdot V}$$

Hence Φ can be thought of as an isomorphism of G-representations that takes $\overline{\bigoplus_{V\in\widehat{G}} \dim V \cdot V}$ to $L^2(G)$. ■

Corollary 2.3.4 (Peter-Weyl)**.** *The irreducible characters of G form an orthonormal basis of the subspace $\mathcal{C}(G)$ of class-functions – in the sense that they span a dense subspace in $\mathcal{C}(G)$.*

Note. Yet another lengthy proof lies ahead.

Proof. Given an irreducible representation V of G consider the algebra $\operatorname{End}(V)$ with

$$T \cdot S = \frac{1}{\dim V} TS$$

and

$$\begin{aligned}
\Psi : \overline{\bigoplus_{V\in\widehat{G}} \operatorname{End}(V)} &\longrightarrow L^2(G) \\
E_{ij} &\longmapsto f_{e_j, e_i}
\end{aligned}$$

We want to establish that Ψ is an isomorphism of algebras between $\overline{\bigoplus_{V\in\widehat{G}} \operatorname{End}(V)}$ and $L^2(G)$ equipped with the convolution product

$$(f_1 * f_2)(g) = \frac{1}{\mu(G)} \int_G f_1(h) f_2(h^{-1}g)\, \mathrm{d}h,$$

This may all seem *extremely arbitrary* – and indeed it is! The point of all this is we'll show that the center of $L^2(G)$ is $\mathcal{C}(G)$ and the center of $\widehat{\bigoplus_{V\in\widehat{G}} \operatorname{End}(V)}$ is precisely what we want it to be. First of all, note that Ψ factors through Φ.

$$\begin{array}{ccc}
\overline{\bigoplus_{V\in\widehat{G}} V^* \boxtimes V} & \xrightarrow{\sim} & \overline{\bigoplus_{V\in\widehat{G}} \operatorname{End}(V)} \\
& \searrow^{\Phi} & \downarrow \Psi \\
& & L^2(G)
\end{array}$$

This implies Ψ is a linear isomorphism. To see that Φ is a homomorphism of algebras, it suffices to observe that

$$\begin{aligned}
(\Psi(E_{ij} \cdot E_{jk}))(g) &= \frac{1}{\dim V}(\Psi(E_{ik}))(g) \\
&= \frac{1}{\dim V} f_{e_k,e_i}(g) \\
&= \frac{1}{\dim V} \langle e_j, e_j \rangle \langle g e_k, e_i \rangle \\
\text{(see theorem 2.1 of [2])} &= \langle f_{e_j,e_i}, f_{e_j,ge_k} \rangle \\
&= \frac{1}{\mu(G)} \int_G \langle h e_j, e_i \rangle \overline{\langle h e_j, g e_k \rangle} \, \mathrm{d}h \\
&= \frac{1}{\mu(G)} \int_G \langle h e_j, e_i \rangle \langle h^{-1} g e_k, e_j \rangle \, \mathrm{d}h \\
&= \frac{1}{\mu(G)} \int_G f_{e_j,e_i}(h) f_{e_k,e_j}(h^{-1}g) \, \mathrm{d}h \\
&= (f_{e_j,e_i} * f_{e_k,e_j})(g) \\
&= (\Psi(E_{ij}) * \Psi(E_{jk}))(g) \\
&\therefore \Psi(T \cdot S) = \Psi(T) * \Psi(S)
\end{aligned}$$

Notice that $\mathcal{C}(G)$ is the center of $L^2(G)$ under the convolution product. Moreover, the center of $\mathrm{End}(V)$ is clearly $\mathbb{C}\,\mathrm{Id}$. We claim that the image of $\mathbb{C}\,\mathrm{Id}$ under Ψ is $\mathbb{C}\chi_V$. Indeed,

$$(\Psi(\mathrm{Id}))(g) = \sum_{i=1}^{n} (\Psi(E_{ii}))(g) = \sum_{i=1}^{n} f_{e_i,e_i}(g) = \chi_V(g)$$

Hence

$$\begin{aligned}
\mathcal{C}(G) &= Z(L^2(G)) \\
&= \Psi\left(Z\left(\overline{\bigoplus_{V \in \widehat{G}} \mathrm{End}(V)} \right)\right) \\
&= \overline{\bigoplus_{V \in \widehat{G}} Z(\Psi(\mathrm{End}(V)))} \\
&= \overline{\bigoplus_{V \in \widehat{G}} \mathbb{C}\chi_V}
\end{aligned}$$

We are done. ■

This entire chapter hopefully stablishes that the representation theory of compact groups is *precisely the same* as the theory of representations of finite groups. Next we will turn our attention to Lie groups and their representations, which substantially deviates from the case of finite groups – Lie groups are *everything but discrete*, so this shouldn't really come as a surprise.

Chapter 3

Smooth Representations of Lie Groups

This chapter is generally based on the second part of *Representation theory: A first course* [9]. We've already discussed Lie groups, but for the uninitiated among you: a Lie group is *a group that is also a smooth manifold* – i.e. a group object in the category **Diff** of smooth manifolds.

$$\begin{array}{ccc} \mathbf{LieGrp} & \longrightarrow & \mathbf{Grp} \\ \big\downarrow & & \big\downarrow \\ \mathbf{Diff} & \longrightarrow & \mathbf{Set} \end{array}$$

Example. The *classical* examples of Lie groups are the *classical groups*, i.e...

(i) The group of invertible $n \times n$ matrices $\mathrm{GL}_n(\mathbb{R})$

(ii) $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$

(iii) The group $\mathrm{SO}(n)$ of orthogonal invertible matrices with real coefficients

(iv) The symplectic group

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ M \in \mathrm{SL}_{2n}(\mathbb{R}) : M \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \right\}$$

All of the groups above are endowed with the Euclidean topology. Since $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous, given $M \in \mathrm{GL}_n(\mathbb{R})$ there should exist some open neighborhood of M completely contained in $\mathrm{GL}_n(\mathbb{R})$. Hence $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ – in particular, it is a closed submanifold of $M_n(\mathbb{R})$. The other groups can be realized as the inverse image of regular values of smooth maps $\mathrm{GL}_n(\mathbb{R}) \longrightarrow M$ – for appropriate M's. It's easy to see that the product in such groups is smooth. Moreover, it follows from Cramer's formula that the inverse map is also smooth. This establishes they all are Lie groups.

This examples are not arbitrary, since it follows from something called *Ado's theorem* that every Lie group is locally isomorphic at the identity to some subgroup of $\mathrm{GL}_n(\mathbb{R})$ for large enough n, but we're getting ahead of ourselves. A morphism of Lie groups $G \longrightarrow H$ is a smooth group homomorphism $G \longrightarrow H$. Again, it doesn't make any sense to forget the geometric structure of given Lie group G when studying it's representations – how else are we supposed to leverage the fact that G is a smooth manifold? This translates to...

Definition. A finite-dimensional representation V of a Lie group G over $\mathbb{R}$ is called *smooth* if

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is a smooth map.

Example 3.0.1. Consider the action of G over G given by conjugation

$$\begin{aligned} \Psi : G &\longrightarrow \mathrm{Inn}(G) \\ g &\longmapsto \Psi(g) : \begin{aligned}[t] G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned} \end{aligned}$$

This action induces a smooth representation $V = T_e G$ where

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto (d\Psi(g))_e \end{aligned}$$

known as *the adjoint representation of G*.

Notice in this case that there is *no* concept of *infinite-dimensional representations of a Lie group*. This is because smooth manifolds are, by definition, finite-dimensional. Hence in general $\mathrm{GL}(V)$ isn't a Lie group for infinite-dimensional V. This is quite convenient for us, since *finite-dimensional stuff* is easier to understand. This implies there's no need to define *smooth intertwining operators*, since every linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth.

Note. Besides the last paragraph, there is something called *geometric analysis*, which allows us to talk about *infinite-dimensional manifolds* and *infinite-dimensional Lie groups*. We won't discuss that in here though.

Our goal is, once again, to classify the (finite-dimensional smooth) representations of a given Lie group up to isomorphism. Just like the *topology* of *topological* groups allows us to effectively study them and their representations, the geometric structure of Lie groups play a vital – and frankly essential – role in this chapter. Surprisingly however, the representation theory of Lie groups is much more *algebraic* than it's topological counter-part.

It's important to note that, in this case, we're really talking about *real* representations of Lie groups – not the usual *complex* representations. This is because we're studying the representation theory of *real* Lie groups, so if we want $\rho : G \longrightarrow \mathrm{GL}(V)$ to be a smooth map, we need $\mathrm{GL}(V)$ to be *real* Lie group. Of course, $\mathrm{GL}_n(\mathbb{C})$ is also a real Lie group, but the distinction between the ground fields complicates our discussion.

Usually one chooses to study representations of a group over $\mathbb{C}$ because of it's rich algebraic and topological structure. This features are usually desired because they tend to play an important part in the solutions to the corresponding classification problems. However, as we'll establish in the following chapters, they are not need in our case. This is one of the many ways in which the representation theory of Lie groups differs from it's finite counterpart.

One could also study the representation theory of *complex Lie groups*, which are, as you might have guessed, *groups that are also complex manifolds*. In this case, the complex Lie group $\mathrm{GL}_n(\mathbb{C})$ serves as a clear analogue to $\mathrm{GL}_n(\mathbb{R})$ is our case – since we'd be interested in *holomorphic representations* or something along these lines. However, holomorphic functions are subject to much greater constraints than smooth maps – for instance, every locally constant holomorphic map is constant. Hence the representation theories of complex Lie groups and real Lie groups are substantially different. We won't discuss the representation theory of complex Lie groups in here.

Back to real Lie groups then… The instrumental peace of the puzzle here is the notion of *Lie algebras*, which can be treated as purely-algebraic structures. We regard Lie algebras primarily as tools for the study of Lie groups, they are not our focus in here. Nevertheless, despite it's title this chapter is mostly dedicated to the study of Lie algebras and their relationship with Lie groups. This is because, as we'll see, the study of representations of Lie algebras is the key to solving our initial classification problem – the one about smooth representations of Lie groups.

3.1 Lie Algebras

In the spirit of dedicating a chapter entitled *Smooth Representations of Lie Groups* to Lie algebras, we'll start a section entitled *Lie Algebras* talking about Lie groups. It turns out Lie groups are *embarrassingly symmetric objects*. Lie *groups* are *groups*, of course. But even more so, the interplay between their geometry and their group structure makes the former tremendously regular and homogeneous – i.e. *Lie groups are also "symmetric" as smooth manifolds* in some vague sense, even though not all Lie groups are *symmetric spaces*. In particular, if G and H are Lie groups it's easy to check that

(i) Given $g \in G$, the translation $\ell_g : G \longrightarrow G$ by g is a diffeomorphism that takes e to g – i.e. Lie groups are homogeneous spaces

(ii) If G is connected and U is a neighborhood of the identity then $G = \langle U \rangle$

(iii) If G and H are simply connected and locally isomorphic at the identity then $G \cong H$

(iv) Given $\phi : G \longrightarrow H$ with $(d\phi)_e$ invertible, ϕ is a covering map

This implies that an invariant vector field $X \in \mathfrak{X}(G)$ – that is, X such that the pushforward $(\ell_g)_* X = (d\ell_g)X = X$ – is uniquely determined by X_e. Hence the space $\mathfrak{g}$ of invariant vector fields over G can be identified with $T_e G$. Now consider the bilinear map

$$\begin{aligned} [\,,] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto XY - YX \end{aligned}$$

where the composition of fields is given point-wise. The map $[\,,]$ is usually called *the Lie bracket on* $\mathfrak{g}$. It's not hard to check that $[\,,]$ is skew-symmetric and satisfies the Jacobian identity

$$[X, [Y, Z]] - [[X, Y], Z] = [[Z, X], Y] \tag{3.1}$$

This brings us to the following definition…

Definition 3.1.1. A Lie algebra $\mathfrak{g}$ over a field k is k-vector space endowed with a skew-symmetric bilinear product $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Jacobian identity (3.1). A homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ of Lie algebras is a linear operator such that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

Clearly, the Lie algebra of invariant vector fields over a Lie group G *is an actual Lie algebra* – which is usually called *the Lie algebra associated with* G. In particular,

Example 3.1.1. The Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ associated with $\mathrm{GL}_n(\mathbb{R})$ can be identified with $T_e \,\mathrm{GL}_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$. Under this identification, the Lie bracket $[\,,] : \mathfrak{gl}_n(\mathbb{R}) \times \mathfrak{gl}_n(\mathbb{R}) \longrightarrow \mathfrak{gl}_n(\mathbb{R})$ is given by

$$[X, Y] = XY - YX,$$

where the product is the usual matrix product.

Definition 3.1.2. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V endowed with a homomorphism of Lie algebras

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

Notice we have defined Lie algebras over an *arbitrary* field. This will come in handy later, when we discuss the *complexification* of a real Lie algebra – this precisely why the fact that $\mathbb{R}$ is not algebraically closed does not matter in our case. For now, however, we're really only interested in the (real) Lie algebra associated with a given Lie group.

This definitions are simple enough, but why are we doing this again? Well, the point is that Lie algebras are invariants: given two Lie groups G and H, if $\mathfrak{g} \cong \mathfrak{h}$ then G and H are locally isomorphic at the identity – i.e. there's a morphism $\Phi : G \longrightarrow H$ and a neighborhood $U \subseteq G$ of the identity such that $\Phi \restriction_U : U \longrightarrow \Phi(U)$ is invertible with smooth inverse.

This allows us to given substance to our claim that *every Lie group is locally isomorphic to some subgroup of* $\mathrm{GL}_n(\mathbb{R})$, whose proof follows directly from Ado's theorem.

Theorem 3.1.1 (Ado)**.** *Every finite-dimensional real Lie algebra is isomorphic to some Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{R})$ *for large enough* n*. In other words,* $\mathfrak{g}$ *admits a faithful representation.*

What we're particularly interested in is that, *in certain contexts*, there is a correspond between morphisms of Lie groups $G \longrightarrow H$ and homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{h}$. In particular, there is a strong correspondence between representations of G and representations of $\mathfrak{g}$ – we'll get to a precise formulation of this soon.

First of all, why should this be true? It may come as a surprise, but in general

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \neq 0$$

In other words, Schwarz's theorem does not hold in arbitrary Lie groups – or indeed arbitrary manifolds. In fact, the Lie bracket $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ measures *the degree to which Schwarz's theorem fails in* G, which is very much part of the geometric structure of G. For instance, the Lie bracket is closely related with the curvature of any given connection in a vector bundle over G – or so says a friend of mine who actually understands what a connection is. This is the whole point of Lie algebras in this context: $\mathfrak{g}$ condenses much of the geometric information of G in an algebraic, easily-manipulatable structure.

I hope that this botched attempt at motivating the definition of a Lie algebra was sufficiently convincing, but that's not really what we're here for – for an alternative, in-depth motivation see [9]. We're here because we want to use representations of Lie algebras to study representations of Lie groups and apparently there's a so called *strong correspondence* between them. What's the correspondence then?

Theorem 3.1.2. *If* G *and* H *are Lie groups with* G *simply connected then there's a one-to-one correspondence between* $\mathrm{Hom}(G, H)$ *and* $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$*. Explicitly, given a morphism* $\Phi : G \longrightarrow H$*,* $\Phi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ *is a homomorphism of Lie algebras and given a homomorphism* $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ *there is a unique morphism* $\Phi : G \longrightarrow H$ *such that* $\varphi = \Phi_*$*.*

In particular,

Corollary 3.1.1. *If* G *is a simply connected Lie group then there is a one-to-one correspondence between representations of* G *and representations of* $\mathfrak{g}$*.*

Proof. It suffices to note that given a vector space V there is a one-to-one correspondence between $\mathrm{Hom}(G, \mathrm{GL}(V))$ and $\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(V))$. ■

Example 3.1.2. The adjoint representation of G induces a representation $V = \mathfrak{g}$ of $\mathfrak{g}$. Surprisingly, this is precisely the "regular" representation

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathfrak{gl}(V) \\ X &\longmapsto \rho(X) : V \longrightarrow V \\ &\qquad\qquad\quad Y \longmapsto [X, Y] \end{aligned}$$

of $\mathfrak{g}$ – see section 2.2 of [5].

In categorical terms, theorem 3.1.2 amounts to saying the functor $\mathrm{Lie} : \mathbf{LieGrp}_{\mathrm{simpl}} \longrightarrow \mathbf{LieAlg}$ – between the category of simply connected Lie groups and the category of finite-dimensional real Lie algebras – that takes G to $\mathfrak{g}$ and a map $G \longrightarrow H$ to it's derivative at the identity is fully-faithful. What's perhaps more surprising is that this functor has a left adjoint – i.e. every finite-dimensional real Lie algebra is the Lie algebra associated with some simply connected Lie group [8], which is known as *Lie's third fundamental theorem.*

Moreover, given a (finite-dimensional real) Lie algebra $\mathfrak{g}$, the simply connected Lie group G such that $\mathfrak{g}$ is the Lie algebra associated with G is unique up to isomorphism. Hence our functor is an equivalence of categories $\mathbf{LieGrp}_{\mathrm{simpl}} \xrightarrow{\sim} \mathbf{LieAlg}$. As interesting as it may sound, however, this functorial formulation of the correspondence is lacking – in the sense that that only G is required to be simply connected for $\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ to hold.

Nevertheless, this correspondence extends to an equivalence between the category of smooth representations of a given (simply connected) Lie group G and that of it's Lie algebra – which takes a representation V of G to the corresponding representation of $\mathfrak{g}$ and an intertwining operator $T : V \longrightarrow W$ to itself. This not only establishes a close connection between representations of Lie groups and algebras, as much as it shows they are essentially the same thing – for simply connected G. Our next goal is to prove theorem 3.1.2, whose essential ingredient is something known as...

3.2 The Exponential Map

Consider the map

$$\begin{aligned} \exp : M_n(\mathbb{R}) &\longrightarrow M_n(\mathbb{R}) \\ X &\longmapsto \operatorname{Id} + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots \end{aligned}$$

This is called *the exponential map*. First of all, notice that

$$\exp(X + Y) = \exp(X)\exp(Y) \tag{3.2}$$

for all $X, Y \in M_n(X)$ with $XY = YX$. In other words, the name exp wasn't chosen at random – the exponential map turns *addition* into *multiplication*, at least in the case where the factors commute. Since $\exp(0) = \operatorname{Id}$, (3.2) implies $\exp(-X) = \exp(X)^{-1}$. Hence exp is really a smooth map $\mathfrak{gl}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R})$.

Moreover, exp is natural in the sense that

$$\begin{array}{ccc} \mathfrak{gl}_n(\mathbb{R}) & \xrightarrow{\Phi_*} & \mathfrak{gl}_m(\mathbb{R}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathrm{GL}_n(\mathbb{R}) & \xrightarrow[\Phi]{} & \mathrm{GL}_m(\mathbb{R}) \end{array}$$

for all morphisms $\Phi : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_m(\mathbb{R})$. Now notice that the derivative of the exponential map at the origin is the identity – in particular, it is a linear isomorphism. Hence the restriction $\exp\restriction_U : U \longrightarrow \exp(U)$ is a diffeomorphism for some neighborhood of U of the origin. It's also important to note that $\operatorname{Id} = \exp(0) \in \exp(U)$.

This is particularly useful for us since it implies that, at least in the connected component of the identity, every morphism $\mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_m(\mathbb{R})$ is determined by it's derivative at the identity. It then follows that *if* $\mathrm{GL}_n(\mathbb{R})$ *was connected, every map* $\mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_m(\mathbb{R})$ *would be determined by it's derivative at the identity*. Unfortunately, $\mathrm{GL}_n(\mathbb{R})$ has precisely two connected components – *almost connected*, but not quite.

One could ask, however, if there is any analogue of the exponential map for some connected G. This is the next step we'll take in the direction of establishing theorem 3.1.2: somehow generalize the exponential map. Given $X \in \mathfrak{gl}_n(\mathbb{R})$, a simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX) = X \cdot \exp(tX) \tag{3.3}$$

Now since $\exp(0) = \operatorname{Id}$, this implies that

$$\begin{aligned} \gamma_X : \mathbb{R} &\longrightarrow \mathrm{GL}_n(\mathbb{R}) \\ t &\longmapsto \exp(tX) \end{aligned}$$

is the integral curve of X that passes trough Id in $t = 0$. In other words, γ_X is the unique curve over $\mathrm{GL}_n(\mathbb{R})$ such that $\dot{\gamma_X}(t) = X \cdot \gamma_X(t)$ and $\gamma_X(0) = \operatorname{Id}$. Hence $\exp(X) = \gamma_X(1)$, where γ_X is the integral curve described above. That's quite a useful alternative to us, since nothing about this is specific to $\mathrm{GL}_n(\mathbb{R})$.

Given a Lie group G and $X \in \mathfrak{g}$, it's not hard to show that there exists a single smooth curve $\gamma : I \longrightarrow G$ such that $\dot{\gamma}(t) = X_{\gamma(t)}$ and $\gamma(0) = e$ – where $I \subseteq \mathbb{R}$ is some (maximal) open interval containing 0. We claim that γ is *local group homomorphism* too – i.e. $\gamma(s + t) = \gamma(s) \cdot \gamma(t)$ for all $s, t \in I$ with $s + t \in I$.

Indeed, if we fix $s \in I$ and define

$$\begin{aligned}\gamma_1(t) &= \gamma(s+t)\\ \gamma_2(t) &= \gamma(s) \cdot \gamma(t)\end{aligned}$$

it's easy to see that

$$\dot{\gamma_1}(t) = X_{\gamma_1(t)} \tag{3.4}$$

$$\dot{\gamma_2}(t) = X_{\gamma_2(t)} \tag{3.5}$$

for all t.

Moreover, $\gamma_1(0) = \gamma_2(0) = \gamma(s)$. It then follows from the uniqueness of the integral curve of a vector field over an arbitrary manifold that $\gamma_1 = \gamma_2$. This allows us to extend γ to the entire real number line, yielding a Lie group morphism $\gamma_X : \mathbb{R} \longrightarrow G$ with $\dot{\gamma_X}(t) = X_{\gamma_X(t)}$ – which is called a *one-parameter subgroup of G*. Lo and behold, we arrive at our general definition…

Definition 3.2.1. The map exponential map on G is given by

$$\begin{aligned}\exp : \mathfrak{g} &\longrightarrow G\\ X &\longmapsto \gamma_X(1)\end{aligned}$$

This extended definition is useful to us not only because it generalizes the exponential map of $\mathrm{GL}_n(\mathbb{R})$ – in the sense that both definitions coincide in this case – but primarily because it preserves most of it's key features. Specifically…

Theorem 3.2.1. *The exponential map is the unique map from $\mathfrak{g}$ to G that takes 0 to e whose derivative at the origin is the identity and whose restriction to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.*

Theorem 3.2.2. *The exponential map is such that*

$$\begin{array}{ccc}\mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h}\\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp}\\ G & \xrightarrow[\Phi]{} & H\end{array}$$

for all $\Phi : G \longrightarrow H$.

Proof. First of all, notice that γ_X is uniquely characterized by the fact that it is a homomorphism and $\dot{\gamma_X}(0) = X_e$. Indeed, given a smooth homomorphism $\gamma : \mathbb{R} \longrightarrow G$ with $\dot{\gamma}(0) = X_e$,

$$\begin{aligned}X_{\gamma(t)} &= (\ell_{\gamma(t)})_* \, X_e\\ &= (d\ell_{\gamma(t)})_e \, \dot{\gamma}(0)\\ \text{(chain rule)} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma(t)\gamma(s)\\ \text{(because } \gamma \text{ is a homomorphism)} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma(t+s)\\ \text{(chain rule)} &= \dot{\gamma}(t+0)\\ &= \dot{\gamma}(t)\end{aligned}$$

In other words, γ is the integral curve γ_X. This implies $\gamma_{\Phi_* X} = \Phi \circ \gamma_X$ – since both of this curves satisfy the conditions above. In particular,

$$\begin{aligned}\Phi(\exp(X)) &= \Phi(\gamma_X(1))\\ &= \gamma_{\Phi_* X}(1)\\ &= \exp(\Phi_* X)\end{aligned}$$

for all $X \in \mathfrak{g}$. ■

As previously mentioned, this last theorem implies...

Corollary 3.2.1. *For connected G, a Lie group morphism $\Phi : G \longrightarrow H$ is determined by it's derivative at the identity Φ_*.*

Proof. Let $\Psi : G \longrightarrow H$ be a morphism such that $\Psi_* = \Phi_*$. We want to establish that $\Psi = \Phi$.

Since the derivative of exp at the origin is the identity, $\exp\restriction_U : U \longrightarrow \exp(U)$ is a diffeomorphism for some open $U \subseteq \mathfrak{g}$ containing the origin – i.e it is a local diffeomorphism at the origin. It then follows from theorem 3.2.2 that $\Phi\restriction_{\exp(U)}$ and $\Psi\restriction_{\exp(U)}$ coincide.

Notice $e \in \exp(U)$ – because the exponential map takes 0 to the identity in G. In particular, $\exp(U)$ is an open neighborhood of e. Now since G is connected, $G = \langle \exp(U) \rangle$, from which we conclude that Φ and Ψ coincide in all of G – given that Φ and Ψ are both group homomorphisms. We are done. ■

An important consequence of theorem 3.2.1 is the fact that (3.3) holds for all G, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) = X_{\exp(tX)}$$

for all $X \in \mathfrak{g}$. Moreover, given a representation V of G,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(tX)) = \rho_* \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) \right) = \rho_*(X)$$

Hence...

Corollary 3.2.2. *Let V and W be representations of G and $T : V \longrightarrow W$ be an intertwining operator . Then T is also an intertwiner with respect to the corresponding representations of $\mathfrak{g}$. In other words,* $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

Proof. Given $X \in \mathfrak{g}$, it suffices to note that (3.3) implies

$$\begin{aligned} X\,T &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)\,T \\ (\text{since } T \text{ is an intertwiner}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T\,\exp(tX) \\ &= T\,X \end{aligned}$$

where $\exp(tX)$ and X stand for their respective actions in V and W. ■

We are now one step closer to establishing our correspondence. Essentially, what we've just proved in the last results is that our map $\mathrm{Hom}(G, H) \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ is injective for connected G – and in particular for simply connected G – and that our map $\mathrm{Hom}_G(V, W) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V, W)$ is well defined. All that's left is to show that these maps are also surjective for simply connected G, which is remarkably simple.

Theorem 3.2.3. *For simply connected G, every homomorphism $\mathfrak{g} \longrightarrow \mathfrak{h}$ is the derivative of some morphism $G \longrightarrow H$ at the identity.*

Proof. Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a homomorphism. Now consider the group $G \times H$, whose Lie algebra is $\mathfrak{g} \oplus \mathfrak{h}$. The condition that φ is a homomorphism is equivalent to the condition that $\mathfrak{j} = \{g \oplus \varphi(g) : g \in \mathfrak{g}\}$ is Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. We claim that there exists some (immersed) subgroup $J \subseteq G \times H$ whose Lie algebra is precisely $\mathfrak{j}$.

If we allow ourselves some leeway and suppose this is the case in a gesture of faith, it's easy to see the derivative of the projection map $\pi_1 : J \longrightarrow G$ at the identity $(\pi_1)_* : \mathfrak{j} \longrightarrow \mathfrak{g}$ is a linear isomorphism. Hence π_1 is isogeny [6] – i.e. J is a covering space with base space G. Now since G is simply connected, π_1 is an isomorphism. It the follows that the composition $\Phi = \pi_2 \circ \pi_1^{-1} : G \longrightarrow H$ is a morphism such that $\Phi_* = \varphi$. ■

Theorem 3.2.4. *Let V and W be representations of G for some connected G and $T : V \longrightarrow W$ be an intertwining operator between the corresponding representations of $\mathfrak{g}$. Then T is also an intertwiner with respect to G. In other words,* $\mathrm{Hom}_{\mathfrak{g}}(V, W) \subseteq \mathrm{Hom}_G(V, W)$.

Proof. Let $X \in \mathfrak{g}$ and define

$$\begin{aligned} \phi_1 : \mathbb{R} &\longrightarrow \mathrm{Hom}(V, W) \\ t &\longmapsto T \exp(tX) \end{aligned}$$

$$\begin{aligned} \phi_2 : \mathbb{R} &\longrightarrow \mathrm{Hom}(V, W) \\ t &\longmapsto \exp(tX)\, T \end{aligned}$$

It then follows from (3.3) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t) &= T\, X \exp(tX) \\ (T \text{ is an intertwiner}) &= X\, T \exp(tX) \\ &= X\, \phi_1(t) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t) &= X \exp(tX)\, T \\ &= X\, \phi_2(t) \end{aligned}$$

Hence ϕ_1 and ϕ_2 are both solutions to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = X\phi(t)$$

with $\phi(0) = T$. It then follows from the existence and uniqueness of solutions of ordinary differential equations that $\phi_1 = \phi_2$. In particular, $\phi_1(1) = \phi_2(1)$, which means $T \exp(X) = \exp(X)\, T$ for all $X \in \mathfrak{g}$. Now since G is connected, $G = \langle \exp(\mathfrak{g}) \rangle$ and therefore

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[T]{} & W \end{array}$$

for all $g \in G$. ■

Problem solved! Here's our correspondence, proved once and for all. Representations of simply connected Lie groups and Lie algebras are, indeed, *the same exact thing*. We are done. Well… not really, are we? We still need to show that the subgroup $J \subseteq G \times H$ of theorem 3.2.4 does, in fact, exist. We're almost there, but once again we'll need the help of some additional tools to finish our proofs, in particular…

3.3 The Campbell-Hausdorff Formula

Our new goal is to prove that…

Theorem 3.3.1. *Let G be a Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra of it's Lie algebra. Then there exists an immersed subgroup $H \subseteq G$ such that $\mathfrak{h}$ is the Lie algebra of H.*

We expect, perhaps naively, that the exponential map $\exp : \mathfrak{h} \longrightarrow H$ is the restriction of $\exp : \mathfrak{g} \longrightarrow G$ to $\mathfrak{h}$. So according to theorem 3.2.1 the one-parameter subgroups of H should be the restriction of $\exp : \mathfrak{g} \longrightarrow G$ to the lines through the origin in $\mathfrak{h}$. Hence a natural candidate is

$$H = \bigcup_{\substack{X \in \mathfrak{h} \\ t}} \exp(tX) = \exp(\mathfrak{h})$$

In other words, H should be the image of the curves over G tangent to the elements of $\mathfrak{h}$. The question now is: is $\exp(\mathfrak{h})$ a subgroup of G? Well, if it is a subgroup, it's Lie algebra is clearly $\mathfrak{h}$. Also, regardless of whether or not it is a subgroup, $\langle \exp(\mathfrak{h}) \rangle$ is. The issue with setting $H = \langle \exp(\mathfrak{h}) \rangle$, however, is that we lose control of it's Lie algebra. So to all intents and purposes it would be quite convenient for us if $\exp(\mathfrak{h}) = \langle \exp(\mathfrak{h}) \rangle$. Fortunately, this is the case, but how do we prove it?

Clearly, $\exp(\mathfrak{h}) \subseteq \langle \exp(\mathfrak{h}) \rangle$. Now given $X, Y \in \mathfrak{h}$, we need to find some $Z \in \mathfrak{h}$ such that $\exp(Z) = \exp(X)\exp(Y)$. Recall that $\exp : \mathfrak{g} \longrightarrow G$ is a local diffeomorphism at the origin. This means that the restriction of the exponential map to some neighborhood U of the origin in $\mathfrak{g}$ is invertible. Let $\log : \exp(U) \longrightarrow U$ be the inverse of exp. Then for $X, Y \in \mathfrak{h}$ close enough of the origin,

$$X * Y = \log(\exp(X)\exp(Y))$$

is such that $\exp(X * Y) = \exp(X)\exp(Y)$.

Given $X \in \mathfrak{h}$, it's easy to check that the integral curve $\gamma_X : \mathbb{R} \longrightarrow G$ passes through $\exp(U)$ at $t = 0$. Hence there should exist some $n > 0$ large enough such that $\gamma_X\left(\frac{1}{n}\right) \in \exp(U)$. By taking $t = \frac{1}{n}$ we arrive at

$$\exp(X) = \gamma_X(1) = \gamma_X(n \cdot t) = \gamma_X(t)^n$$

This implies $\langle \exp(U \cap \mathfrak{h}) \rangle = \langle \exp(\mathfrak{h}) \rangle$, so it suffices to show that $\exp(\mathfrak{h})$ is locally closed under multiplication – i.e. $X * Y \in \mathfrak{h}$ for all $X, Y \in U \cap \mathfrak{h}$. We'll, again, begin by studying a simpler case: the case of $\mathfrak{gl}_n(\mathbb{R})$. In this case, a simple calculation shows

$$\log(M) = (M - \mathrm{Id}) - \frac{(M - \mathrm{Id})^2}{2} + \frac{(M - \mathrm{Id})^3}{3} - \dots,$$

which converges only for M sufficiently close to Id.

Now by expanding

$$\begin{aligned} \exp(X)\exp(Y) &= \left(\mathrm{Id} + X + \frac{X^2}{2} + \cdots\right)\left(\mathrm{Id} + Y + \frac{Y^2}{2} + \cdots\right) \\ &= \mathrm{Id} + (X + Y) + \left(\frac{X^2}{2} XY + \frac{Y^2}{2}\right) + \cdots \end{aligned}$$

we arrive at

$$\begin{aligned} X * Y &= (X + Y) + \left(-\frac{(X+Y)^2}{2} + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right)\right) + \cdots \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \end{aligned} \tag{3.6}$$

This calculations are, of course, quite trivial, as long as you simply copy them from a book without even bothering to check the second term like I did. Anyway, the point of (3.6) isn't it's proof or the precise formula of the series in the right site: the point is that each term in the right side of the last equality is a scalar multiple of a Lie bracket. This implies that $X * Y \in \mathfrak{h}$ for $X, Y \in \mathfrak{h} \subseteq \mathfrak{gl}_n(\mathbb{R})$ sufficiently close to the origin.

Equation (3.6) is called *the Campbell-Hausdorff formula*, and it's proof is nothing short of a painful exercise in real analysis as far as I can tell – we won't include it in here and neither does the book I copied it from [9]. Now given an arbitrary Lie group G, remember that G is locally isomorphic to some subgroup of $\mathrm{GL}_n(\mathbb{R})$ at the identity for n large enough. Hence the Campbell-Hausdorff formula holds for *all* Lie groups. We are essentially done.

Proof of Theorem 3.3.1. Let $H = \exp(\mathfrak{h})$. Then given $X, Y \in \mathfrak{h}$ sufficiently close to the origin,

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots \in \mathfrak{h}$$

since $\mathfrak{h}$ is closed under the Lie bracket and the limit of sequences – every subspace of a finite-dimensional space is closed. So there exists $Z \in \mathfrak{h}$ such that $\exp(Z) = \exp(X)\exp(Y)$ – given by $Z = X * Y$. This implies $\langle H \rangle \subseteq H$, which is to say, H is a subgroup of G. ■

Notice that Lie's third fundamental theorem also follows from theorem 3.3.1. Indeed, given a finite-dimensional real Lie algebra $\mathfrak{g}$, Ado's theorem and theorem 3.3.1 imply that there exists some subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$ such that $\mathfrak{g}$ is the Lie algebra of G – explicitly, $G = \exp(\mathfrak{g})$. By taking the simply connected cover of G we arrive at the desired conclusion – see [8] for further details.

We've spoken very little about representations so for. Now that we've established our correspondence, we'll carry out our initial goal of *classifying the smooth representations of simply connected Lie groups* by studying the representations of finite-dimensional real Lie algebras.

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